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On the energy increase in space-collapse models

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Abstract
A typical feature of spontaneous collapse models which aim at localizing wavefunctions in space is the violation of the principle of energy conservation. In the models proposed in the literature, the stochastic field which is responsible for the localization mechanism causes the momentum to behave like a Brownian motion, whose larger and larger fluctuations show up as a steady increase of the energy of the system. In spite of the fact that, in all situations, such an increase is small and practically undetectable, it is an undesirable feature that the energy of physical systems is not conserved but increases constantly in time, diverging for $t \to \infty$. In this paper, we show that this property of collapse models can be modified: we propose a model of spontaneous wavefunction collapse sharing all most important features of usual models but such that the energy of isolated systems reaches an asymptotic finite value instead of increasing with a steady rate.

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1. Introduction

As is well known, space-collapse models [1–14] aim at a solution of the macro-objectification or measurement problem in quantum mechanics by suitably modifying the Schrödinger equation with non-linear stochastic terms. One of the characteristic features of these models is the violation of energy conservation for isolated systems; such a violation is determined by the stochastic process responsible for the localization mechanism, which induces larger and larger fluctuations of the wavefunction in the momentum space [4]: these increasing fluctuations, in turn, determine the increase of the energy of the system [15]. For typical values of the parameters, such an increase is very small and undetectable with present-day technology [1]; still, one would wish to restore the principle of energy conservation within space-collapse models.

In this paper, we make one step towards this direction: we analyse a model of wavefunction space collapse for which the energy of isolated systems does not increase indefinitely, but reaches an asymptotic finite value. An analogy with quantum Brownian motion will show that the stochastic process acts like a dissipative medium which thermalizes the system to a fixed temperature (the temperature of the medium) and will suggest how to restore perfect energy conservation.

The paper is organized as follows: after a brief review of the main features of dynamical reduction models (section 2), we introduce the collapse model which is the subject of the paper (section 3). In section 4, we study the master equation for the statistical operator originating from the stochastic dynamics: this will provide the rationale for the choice of the localization operator which defines the model. In sections 5–8, we will study in detail the most relevant properties of the model: we will analyse the time evolution of Gaussian wavefunctions (section 5), the collapse mechanism and collapse probability (section 6); we will see that the physical predictions of the model agree with very high accuracy with standard quantum-mechanical predictions and, at the same time, the model reproduces classical mechanics at the macroscopic level (section 7). We will finally discuss the issue related to energy non-conservation (section 8) and conclude with some final remarks (section 9).

2. Structure of dynamical reduction models

The typical structure of the evolution equation of collapse models is[7]

$$\mathrm{d}\psi_t = \left[-\frac{\mathrm{i}}{\hbar} H\,\mathrm{d}t + \sqrt{\lambda}(A - r_t)\,\mathrm{d}W_t - \frac{\lambda}{2}\left(A^\dagger A - 2r_t A + r_t^2\right)\mathrm{d}t \right]\psi_t, \tag{1}$$

with

$$r_t = \tfrac{1}{2}\langle\psi_t|(A + A^\dagger)|\psi_t\rangle. \tag{2}$$

The operator H is related to the standard quantum Hamiltonian of the system, while A is the *reduction operator*, i.e., the operator on whose eigenmanifolds one wants to reduce the statevector, as a consequence of the collapse mechanism; the positive constant λ sets the strength of the collapse mechanism. The stochastic dynamics is governed by a standard Wiener process W_t defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that the equation is non-linear but preserves the norm of the statevector.

In the literature on collapse models, the operator A is usually assumed to be self-adjoint; in such a case, and if one further assumes that it has only a discrete spectrum, it can be proven [2] that the form of the second and third terms of equation (1), which modify the standard Schrödinger evolution, is such that

(1) The statevector collapses with respect to the 'preferred basis' generated by the operator A, i.e., for almost all realizations of the stochastic process there exists an eigenstate $|a_n\rangle$ of A (depending of course on the realization of the stochastic process) such that

$$|\psi_t\rangle \longrightarrow |a_n\rangle, \qquad \text{for} \quad t \to \infty. \tag{3}$$

[7] Of course, this form can be generalized, e.g. by adding a finite (or countable) number of operators A_i, each of which is coupled to a Wiener process W_i. Moreover, the Wiener processes may be complex instead of real, as assumed here.

(2) The average $\mathbb{E}[\langle P \rangle_t] \equiv \mathbb{E}[\langle \psi_t | P | \psi_t \rangle]$ of any operator P which commutes with A is constant in time, i.e., $\mathbb{E}[\langle \psi_t | P | \psi_t \rangle] = \langle \psi_0 | P | \psi_0 \rangle$. In particular, if P is a projection operator relative to an eigenmanifold of A, this together with property (3) implies that the probability for the statevector to be reduced into the eigenmanifold associated with P is equal to $\langle \psi_0 | P | \psi_0 \rangle$, i.e., to the probability that standard quantum mechanics associates with the collapse, as a result of a measurement of the operator A. This is due to the fact that $\langle \psi_t | P | \psi_t \rangle$ turns out to be a martingale, thanks to the particular structure of equation (1), so that by the martingale property $\mathbb{E}[\langle \psi_t | P | \psi_t \rangle] = \langle \psi_0 | P | \psi_0 \rangle$ [8].

It is important to keep in mind that the above results are valid only when the standard Hamiltonian H either commutes with A or is equal to zero; in all other cases, such results are only approximate, the approximation depending on the value of λ.

3. The model

In the literature, A has been mainly taken equal to the position operator q, or a function of q like in the continuous version [2] of the original GRW model [1], the reason being that the operator q is the most natural candidate for localizing wavefunctions in space. As anticipated in the previous section, one consequence of such a choice is that the energy of the system increases in time, diverging for time going to infinity; it is then natural to wonder whether a different choice for A can preserve all most important features of collapse models, but at the same time cure this energy non-conservation. This problem finds a partly positive solution by making the following ansatz[8] for A:

$$A = q + \mathrm{i}\frac{\alpha}{\hbar}p, \tag{4}$$

where p is the momentum operator. Moreover, we define the operator H as follows:

$$H = H_0 + \frac{\lambda\alpha}{2}\{q, p\}, \tag{5}$$

where H_0 is the standard quantum Hamiltonian. In the following sections we will analyse the most relevant physical properties of the model and we will focus our attention to the case of a *free* particle: $H_0 = p^2/2m$, where m is the mass of the particle.

The model is defined in terms of the two constants λ and α; for reasons which will be clear in the following, we will assume them to vary with the mass of the particle as follows:

$$\lambda = \frac{m}{m_0}\lambda_0, \qquad \alpha = \frac{m_0}{m}\alpha_0, \tag{6}$$

where m_0 is a reference mass which we choose to be equal to that of a nucleon while λ_0 and α_0 are fixed constants which we take equal to

$$\lambda_0 \simeq 10^{-2}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1}, \tag{7}$$

$$\alpha_0 \simeq 10^{-18}\ \mathrm{m}^2. \tag{8}$$

[8] A localization operator involving q and p has also been considered in [16] (note however that the form of the localization operator is different from ours) but with a different aim, i.e., that of studying whether the presence of a p term instead of only a q term can improve the localization mechanism. The authors prove that, for any physically reasonable choice of the parameters of their model, such term does not affect in an appreciable way the collapse mechanism. Here we show that a p term is important as it can change the time evolution of the mean energy, avoiding it to increase constantly in time. The authors of [14] analyse a stochastic differential equation similar to our equation (1) where both a q and a p term are present: they mainly focus their attention on the application of the formalism to the theory of open quantum systems and decoherent histories. One of their main results is a localization theorem which we will apply to our model to prove the collapse of wavefunctions to localized states.

As will be shown in section 7, this numerical choice for the parameters guarantees that the model reproduces almost exactly the physical predictions of standard quantum mechanics at the microscopic level and reproduces classical mechanics at the macroscopic level.

Before concluding this section, we note that, in order to find the solutions of equation (1) and to study their properties, it is convenient to consider also a linearized version of equation (1) [2, 3]:

$$\mathrm{d}\phi_t(x) = \left[-\frac{\mathrm{i}}{\hbar} H\,\mathrm{d}t + \sqrt{\lambda} A\,\mathrm{d}\xi_t - \frac{\lambda}{2} A^\dagger A\,\mathrm{d}t\right]\phi_t(x), \tag{9}$$

where ξ_t is a standard Wiener processes defined on a new probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. In [4], the relation between the probability measures $\mathbb{Q}$ and $\mathbb{P}$ and the relation between the stochastic processes W_t and ξ_t are discussed. Here we simply recall how one can use the above linear equation to find a solution of equation (1):

(i) Find the solution ϕ_t of equation (9), with the initial condition $\phi_0 = \psi_0$.
(ii) Normalize the solution[9]: $\phi_t \to \psi_t = \phi_t/\|\phi_t\|$.
(iii) Make the substitution: $\mathrm{d}\xi_t \to \mathrm{d}W_t = \mathrm{d}\xi_t - 2\sqrt{\lambda} r_t$.

The wavefunction ψ_t thus obtained is a solution of equation (1).

4. The master equation for the statistical operator

In order to better understand the modifications to the Schrödinger dynamics induced by equation (1) and the motivations for the precise choice of its form, apart from the martingale structure, and in particular in order to see why the choice (4) for A can partially cure the problem of the energy increase, it is worthwhile considering the related equation for the statistical operator $\rho_t \equiv \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$, which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{\hbar}[H_0, \rho_t] - \frac{\lambda}{2}[q, [q, \rho_t]] - \frac{\lambda\alpha^2}{2\hbar^2}[p, [p, \rho_t]] - \mathrm{i}\frac{\lambda\alpha}{\hbar}[q, \{p, \rho_t\}], \tag{10}$$

that is the typical structure of master equation for the quantum description of Brownian motion, where both friction and diffusion are taken into account and positivity of the statistical operator is granted at all times. The obvious difference between equation (10) and the master equation for quantum Brownian motion lies in the meaning of the coefficients, here related to the two fundamental constants of the model λ and α, rather than to the friction coefficient and the temperature of the bath. The quantum Brownian motion master equation is in fact given by [17–19]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{\hbar}[H_0, \rho_t] - \gamma\frac{2M}{\beta\hbar^2}[q, [q, \rho_t]] - \gamma\frac{\beta}{8M}[p, [p, \rho_t]] - \frac{\mathrm{i}}{\hbar}\gamma[q, \{p, \rho_t\}], \tag{11}$$

where γ is the friction coefficient and β is the inverse temperature of the background medium; the second and third terms at rhs account for diffusion, with coefficients proportional to the squared thermal wavelength $\Delta x_{\mathrm{th}}^2 = \beta\hbar^2/4M$ and the squared thermal momentum spread $\Delta p_{\mathrm{th}}^2 = M/\beta$ satisfying the minimum uncertainty relation $\Delta p_{\mathrm{th}}\Delta x_{\mathrm{th}} = \hbar/2$, while the last is due to friction and ensures that the energy of the test particle asymptotically goes to the equipartition value depending only on the temperature of the bath. Note that in the quantum description, friction, which accounts for the finite energy growth, is of necessity related to diffusion in order to preserve the Heisenberg uncertainty relation [17, 20, 21]. A fundamental result, in order to understand how equation (10) and therefore the striking similarity with

[9] If $\|\phi_t\| = 0$, then one can set ψ_t equal to any fixed unitary vector.

quantum Brownian motion appears, is Holevo's characterization of translation-covariant generators of quantum-dynamical semigroups [22], according to which further important restrictions can be put on the operators appearing in the general so-called Lindblad structure, once symmetry under translations is taken into account. In fact, according to Holevo's result, if the generator of the dynamics $\mathcal{L}$ is translation covariant, i.e., commutes with the action of the unitary representation of translations $U(a) = \exp[-(\mathrm{i}/\hbar)ap]$

$$\mathcal{L}[U(a)\rho U^{\dagger}(a)] = U(a)\mathcal{L}[\rho]U^{\dagger}(a) \tag{12}$$

for all real a, then its general structure, given that q appears at most quadratically, is the following:

$$\mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}[H(p), \rho] + \mathcal{L}_{\mathrm{G}}[\rho], \tag{13}$$

with $H(p)$ a self-adjoint operator only depending on the momentum operator p and $\mathcal{L}_{\mathrm{G}}$ (where G stands for Gaussian) is given by

$$\mathcal{L}_{\mathrm{G}}[\rho] = -\frac{\mathrm{i}}{\hbar}[a_0 q + H_{\mathrm{eff}}(q, p), \rho] + \left[K\rho K^{\dagger} - \frac{1}{2}\{K^{\dagger}K, \rho\}\right], \tag{14}$$

with

$$K = a_1 q + L_1(p), \qquad a_0, a_1 \in \mathbb{R},$$

$$H_{\mathrm{eff}}(q, p) = \frac{\hbar}{2\mathrm{i}} a_1 \big(q L_1(p) - L_1^{\dagger}(p) q\big)$$

and $L_1(p)$ a generally complex function of its argument. The requirement of translational invariance is a natural and compelling one for dynamical reduction models, since the modification of quantum mechanics by a universal noise should by no way break the homogeneity of space, introducing some preferred space location. The restriction to mappings at most quadratic in the position operator q is linked to the fact that we are looking for a generalization of the most simple dynamical reduction model where $A = q$ and the associated master equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{\hbar}[H_0, \rho_t] - \frac{\lambda}{2}[q, [q, \rho_t]], \tag{15}$$

often considered in the literature (see e.g. [4] and references therein) even though leading to a steady energy increase. In view of equation (14), the most straightforward extension of equation (15) including a friction term proportional to velocity is obtained setting $a_0 = 0$, i.e., no external constant force since we are considering the modification to Schrödinger dynamics for a free particle, $a_1 = \sqrt{\lambda}$ and $L_1(p) = \mathrm{i}\sqrt{\lambda}(\alpha/\hbar)p$, thus directly obtaining the ansatz given in equation (4). With this choice of functions and parameters, equation (13) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{\hbar}\left[H_0 + \frac{\lambda\alpha}{2}\{q, p\}, \rho_t\right] + \lambda\left[A\rho A^{\dagger} - \frac{1}{2}\{A^{\dagger}A, \rho\}\right], \tag{16}$$

with $A = q + \mathrm{i}(\alpha/\hbar)p$, as in equation (4), which is immediately seen to be equivalent to equation (10), thus giving the rationale for our choice for the operator A.

Note that looking at equation (16), one might erroneously be led to think that the modification to Schrödinger dynamics amounts to a change in the Hamiltonian plus a correction due to a universal noise given by a mapping in the Lindblad form with a single so-called Lindblad operator A. This standpoint, implicitly adopted in [14], and which has often led to confusion [23], is actually misleading. The result by Lindblad, which is strictly speaking only valid when the generator is bounded, but as shown by Lindblad himself [20] and by the quoted results of Holevo also holds for the case at hand, states that the generator of a completely positive quantum-dynamical semigroup has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{\hbar}[H,\rho_t] + \sum_i \left[L_i\rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i,\rho\}\right], \tag{17}$$

where the self-adjoint operator H is not necessarily the free Hamiltonian of the system, giving its dynamics when it is not coupled to the environment or some noise source. In contrast, it usually happens, e.g. when the Lindblad structure appears in the reduced description of a system coupled to some reservoir, that in the commutator at rhs of (17) an operator appears, which is the sum of the free Hamiltonian and some other self-adjoint operator, this other contribution disappearing together with the rest of the Lindblad form when the coupling vanishes, as it correctly happens in equation (16) if the fundamental constant λ is set to zero. The general structure (17) cannot be thought of as being made up of two distinct parts, since the Lindblad characterization pertains to the structure as a whole. Note however that the free Hamiltonian can still be put into evidence in (1) according to

$$\mathrm{d}\psi_t = \left[-\frac{\mathrm{i}}{\hbar}H_0\,\mathrm{d}t + \sqrt{\lambda}(A-r_t)\,\mathrm{d}W_t - \frac{\lambda}{2}\left(A^\dagger A - 2r_tA + r_t^2 + \frac{1}{2}(A^2 - A^{\dagger 2})\right)\mathrm{d}t\right]\psi_t, \tag{18}$$

even though in this equivalent expression the martingale structure is less evident.

5. Single-Gaussian solution

Gaussian wavefunctions are very special and they are often used to represent typical physical situations; we now show that, as for the standard Schrödinger equation, our stochastic equation preserves the form of Gaussian wavefunctions and, at the same time, we analyse their evolution in time. Let us then consider the following class of wavefunctions:

$$\phi_t(x) = \exp[-a_t(x-\overline{x}_t)^2 + \mathrm{i}\overline{k}_t x + \gamma_t], \tag{19}$$

where a_t and γ_t are complex functions of time, while $\overline{x}_t$ and $\overline{k}_t$ are real. By following the procedure outlined in [4], one can show that the above parameters obey the following stochastic differential equations[10]:

$$\mathrm{d}a_t = \left[-\frac{2\mathrm{i}\hbar}{m}a_t^2 - 4\lambda\alpha a_t + \lambda\right]\mathrm{d}t, \tag{20}$$

$$\mathrm{d}\overline{x}_t = \frac{\hbar}{m}\overline{k}_t\,\mathrm{d}t + \sqrt{\lambda}\left[\frac{1}{2a_t^{\mathrm{R}}} - \alpha\right]\mathrm{d}W_t, \tag{21}$$

$$\mathrm{d}\overline{k}_t = -2\lambda\alpha\overline{k}_t\,\mathrm{d}t - \sqrt{\lambda}\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\,\mathrm{d}W_t. \tag{22}$$

We have omitted the equation for γ_t since the real part can be absorbed into the normalization factor, while the imaginary part represents an irrelevant global phase.

5.1. The time evolution of a_t

The parameter a_t is particularly important since it is related to the spread of the wavefunction (19) in position and momentum, according to the following expressions:

$$\sigma_q(t) \equiv \sqrt{\langle q^2\rangle - \langle q\rangle^2} = \frac{1}{2}\sqrt{\frac{1}{a_t^{\mathrm{R}}}}, \qquad \sigma_p(t) \equiv \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \hbar\sqrt{\frac{(a_t^{\mathrm{R}})^2 + (a_t^{\mathrm{I}})^2}{a_t^{\mathrm{R}}}}, \tag{23}$$

[10] The superscripts 'R' and 'I' denote, respectively, the real and imaginary parts of the corresponding quantities.

equation (20) for a_t can be easily solved; one gets

$$a_t = -\frac{1}{2}\left[A + \mathrm{i}B\tanh\left(\frac{\hbar}{m}Bt + k\right)\right], \tag{24}$$

with

$$A = -2\mathrm{i}\frac{\lambda\alpha m}{\hbar}, \qquad B = \sqrt{\frac{4\lambda^2\alpha^2 m^2}{\hbar^2} + \mathrm{i}\frac{2\lambda m}{\hbar}}; \tag{25}$$

the constant k sets the initial condition a_0.

After a tedious calculation, one can write explicitly the time evolution of the real and imaginary parts[11] of a_t:

$$a_t^{\mathrm{R}} = \frac{m\omega}{2\sqrt{2}\hbar}\frac{\sin\theta\sinh(\omega_1 t + \varphi_1) + \cos\theta\sin(\omega_2 t + \varphi_2)}{\cosh(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi_2)}, \tag{26}$$

$$a_t^{\mathrm{I}} = \frac{-m\omega}{2\sqrt{2}\hbar}\left[\frac{\cos\theta\sinh(\omega_1 t + \varphi_1) - \sin\theta\sin(\omega_2 t + \varphi_2)}{\cosh(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi_2)} - \frac{2\sqrt{2}\lambda\alpha}{\omega}\right], \tag{27}$$

where we have introduced the following two frequencies:

$$\omega_1 = \sqrt{2}\omega\cos\theta, \qquad \omega_2 = \sqrt{2}\omega\sin\theta, \tag{28}$$

the frequency ω and the angle θ being defined as follows:

$$\omega = 2\sqrt[4]{4\lambda_0^4\alpha_0^4 + \frac{\lambda_0^2\hbar^2}{m_0^2}} \simeq 10^{-5}\ \mathrm{s}^{-1}, \tag{29}$$

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{\hbar}{2\lambda_0\alpha_0^2 m_0}\right] \simeq \frac{\pi}{4}; \tag{30}$$

note that, due to the specific dependence of both λ and α on m as given by equation (6), both ω and θ are independent of the mass of the particle, and thus are two constants of the model. Note also that—as is easy to prove—if $a_0^{\mathrm{R}} > 0$, then $a_t^{\mathrm{R}} > 0$ for any subsequent time t: this implies that Gaussian solutions do not diverge in time.

5.2. The spread in position and momentum

According to (23), the spread in position of the Gaussian wavefunction (19) evolves in time as follows[12]:

$$\sigma_q(t) = \sqrt{\frac{\hbar}{\sqrt{2}m\omega}\frac{\cosh(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi_2)}{\sin\theta\sinh(\omega_1 t + \varphi_1) + \cos\theta\sin(\omega_2 t + \varphi_2)}}. \tag{31}$$

Here we can see one of the effects of the reduction mechanism: while in the standard quantum case the spread (in position) of the wavefunction of a free quantum particle increases in time, diverging for $t \to \infty$, the spread according to our model reaches the asymptotic *finite* value ('kg' stands for kilogram and 'm' for metre)

$$\overline{\sigma}_q \equiv \sigma_q(\infty) = \sqrt{\frac{\hbar}{\sqrt{2}m\omega\sin\theta}} \simeq \left(10^{-15}\sqrt{\frac{\mathrm{kg}}{m}}\right)\mathrm{m}. \tag{32}$$

[11] The two parameters φ_1 and φ_2 are functions of the initial conditions.

[12] Note that the evolution of $\sigma_q(t)$ (and also of $\sigma_p(t)$) is *deterministic* and depends on the noise W_t only indirectly, through the constant λ.

The asymptotic spread decreases for increasing values of the mass of the particle, this property entailing that, as we shall discuss in more detail in section 7, wavefunctions of macroscopic objects are almost always very well localized in space, so well that they practically behave like point-like particles.

The time evolution for $\sigma_p(t)$ can also be obtained from (23) and, as it happens for the spread in position, also the spread in momentum asymptotically reaches a finite value, which is

$$\overline{\sigma}_p \equiv \sigma_p(\infty) = \sqrt{\frac{\hbar m \omega}{2\sqrt{2}} \frac{\sin^2\theta + (\cos\theta - \kappa)^2}{\sin\theta}}$$
$$\simeq \left(10^{-19}\sqrt{\frac{m}{\mathrm{kg}}}\right)\frac{\mathrm{kg\ m}}{\mathrm{s}}, \tag{33}$$

where

$$\kappa = \frac{2\sqrt{2}\lambda_0\alpha_0}{\omega} \simeq 10^{-14}. \tag{34}$$

To conclude the section, we compute the product of the two asymptotic spreads

$$\overline{\sigma}_q \cdot \overline{\sigma}_p = \frac{\hbar}{2}\sqrt{1 + \frac{(\cos\theta - \kappa)^2}{\sin^2\theta}}, \tag{35}$$

which is almost equal to $\hbar/\sqrt{2}$, with our choice (7) and (8) for the parameters. Note however that for any choice of λ_0 and α_0, Heisenberg uncertainty relations are fulfilled.

In accordance with [24], any Gaussian solution having these asymptotic values for the spread in position and momentum will be called a 'stationary solution' of equation (1). Of course, the term 'stationary' does not mean that such wavefunctions do not evolve in time; as a matter of fact (see the following discussion), they always undergo a random motion both in position and momentum which never stops. The term 'stationary' refers only to the shape of the wavefunction: stationary solutions are special wavefunction which are *Gaussian* and with a *fixed* spread in position and momentum, given by equations (32) and (33).

5.3. The mean in position and momentum

The mean $\langle q\rangle_t$ in position of the wavefunction and the mean $\langle p\rangle_t$ in momentum satisfy the following stochastic differential equations which can be derived from equations (21) and (22):

$$\mathrm{d}\langle q\rangle_t = \frac{1}{m}\langle p\rangle_t\,\mathrm{d}t + \sqrt{\lambda}\left[\frac{1}{2a_t^{\mathrm{R}}} - \alpha\right]\mathrm{d}W_t, \tag{36}$$

$$\mathrm{d}\langle p\rangle_t = -2\lambda\alpha\langle p\rangle_t\,\mathrm{d}t - \sqrt{\lambda}\hbar\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\,\mathrm{d}W_t. \tag{37}$$

Their average values evolve as follows:

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\langle q\rangle_t] = \mathbb{E}[\langle p\rangle_t], \tag{38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\langle p\rangle_t] = -2\lambda\alpha\mathbb{E}[\langle p\rangle_t]. \tag{39}$$

The first equation reproduces the classical relation between position and momentum of a particle while the second equation predicts that the momentum decays exponentially in time:

$$\mathbb{E}[\langle p\rangle_t] = \langle p\rangle_0\,\mathrm{e}^{-2\lambda\alpha t}, \tag{40}$$

with

$$2\lambda\alpha = 2\lambda_0\alpha_0 \simeq 10^{-20}\ \mathrm{s}^{-1}, \tag{41}$$

which is an extremely slow decay rate, not depending on the mass of the particle.

For completeness, we consider also the covariance matrix

$$C(t) = \mathbb{E}\left[\begin{bmatrix}\langle q\rangle_t - \mathbb{E}[\langle q\rangle_t]\\ \langle p\rangle_t - \mathbb{E}[\langle p\rangle_t]\end{bmatrix}\cdot\begin{bmatrix}\langle q\rangle_t - \mathbb{E}[\langle q\rangle_t]\\ \langle p\rangle_t - \mathbb{E}[\langle p\rangle_t]\end{bmatrix}^{\top}\right]$$
$$\equiv \begin{bmatrix} C_{q^2}(t)C_{qp}(t)\\ C_{pq}(t)C_{p^2}(t)\end{bmatrix},$$

whose coefficients satisfy the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{q^2}(t) = \frac{2}{m}C_{qp}(t) + \lambda\left(\frac{1}{2a_t^{\mathrm{R}}} - \alpha\right)^2, \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{qp}(t) = \frac{1}{m}C_{p^2}(t) - 2\lambda\alpha C_{qp}(t) - \lambda\hbar\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\left(\frac{1}{2a_t^{\mathrm{R}}} - \alpha\right), \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{p^2}(t) = -4\lambda\alpha C_{p^2}(t) + \lambda\hbar^2\left(\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\right)^2. \tag{44}$$

In section 7.2, we will discuss the physical implications of the above equations in connection with the dynamics of macroscopic objects.

6. Asymptotic behaviour of the general solution

In the previous section, we have seen that any Gaussian solution converges towards a stationary solution, i.e., towards a Gaussian wavefunction with a fixed finite value both for the spread in position and momentum, given by equations (32) and (33). In this section, we prove that the spread $\sigma_q(t)$ of *any* wavefunction converges with probability 1 towards $\overline{\sigma}_q$: this means that any initial wavefunction converges to a localized solution; for the proof we will follow the same strategy as [14].

6.1. The reduction process

It is easy to see that a Gaussian stationary solution is an eigenstate of the operator

$$O = p - 2\mathrm{i}\hbar a_\infty q, \tag{45}$$

where

$$a_\infty = \frac{m\omega}{2\sqrt{2}\hbar}[\sin\theta - \mathrm{i}(\cos\theta - \kappa)]; \tag{46}$$

the proof basically consists in showing that the variance

$$\sigma_O^2(t) \equiv \langle\psi_t|[O^\dagger - \langle O^\dagger\rangle][O - \langle O\rangle]|\psi_t\rangle \tag{47}$$

of the operator O vanishes for $t \to \infty$.

The first step is to rewrite $\sigma_O^2(t)$ in terms of the variances associated with the operators q and p:

$$\sigma_O^2(t) = \sigma_p^2(t) + \frac{\overline{\sigma}_p^2}{\overline{\sigma}_q^2}\sigma_q^2(t) - 2\frac{\overline{\sigma}_{q,p}^2}{\overline{\sigma}_q^2}\sigma_{q,p}^2(t) - \frac{\hbar^2}{2\overline{\sigma}_q^2}, \tag{48}$$

where we have defined

$$\sigma_{q,p}^2(t) = \tfrac{1}{2}[\langle\psi_t|[q - \langle q\rangle][p - \langle p\rangle]|\psi_t\rangle + \langle\psi_t|[p - \langle p\rangle][q - \langle q\rangle]|\psi_t\rangle], \quad (49)$$

so that for a Gaussian wavefunction such as (19)

$$\sigma_{q,p}(t) = \sqrt{-\frac{\hbar}{2}\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}} \quad (50)$$

and $\overline{\sigma}_{q,p}$ corresponds to the value of $\sigma_{q,p}(t)$ when the state $|\psi_t\rangle$ is a stationary Gaussian solution.

After a rather long calculation (see appendix A for the details), it is possible to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\sigma_O^2(t)\big] = -4\lambda\mathbb{E}\left[\overline{\sigma}_q^2\sigma_O^2(t) + \overline{\sigma}_{q,p}^4\left(\frac{\sigma_q^2(t)}{\overline{\sigma}_q^2} - \frac{\sigma_{q,p}^2(t)}{\overline{\sigma}_{q,p}^2}\right)^2 + \frac{\hbar^2}{4\overline{\sigma}_q^4}\big(\sigma_q^2(t) - \overline{\sigma}_q^2\big)^2\right] \leqslant 0. \quad (51)$$

Since $\sigma_O^2(t)$ is by definition a positive quantity, the above equation is consistent if and only if the rhs vanishes for any $\omega \in \Omega$, with the possible exception of a subset of measure 0. This in particular implies both that $\sigma_O^2(t) \to 0$ a.s. and that $\sigma_q(t) \to \overline{\sigma}_q$ a.s., which is the desired result, i.e., the wavefunction converges to a localized solution.

6.2. The localization probability

Once proved that equation (1) with the choice (4) for the operator A induces the localization of the wavefunction in space, it becomes natural to analyse the probability for a localization to occur within a given interval of the real axis. Such an analysis can be developed along the same lines as [4].

Let us consider the probability measure

$$\mu_t(\Delta) \equiv \mathbb{E}_{\mathbb{P}}[\|P_\Delta\psi_t\|^2], \quad (52)$$

defined on the Borel sigma algebra $\mathcal{B}(\mathbb{R})$ of the real axis, where P_Δ is the projection operator associated with the Borel subset Δ of $\mathbb{R}$; such a measure is identified by the density $p_t(x) \equiv \mathbb{E}_{\mathbb{P}}[|\psi_t(x)|^2]$:

$$\mu_t(\Delta) = \int_\Delta p_t(x)\,\mathrm{d}x. \quad (53)$$

The density $p_t(x)$ corresponds to the diagonal element $\langle x|\rho_t|x\rangle$ of the statistical operator $\rho_t \equiv \mathbb{E}_{\mathbb{P}}[|\psi_t\rangle\langle\psi_t|]$, solution of the master equation (10). In appendix B, we show how the general solution of this master equation in the position representation can be obtained; the final expression, as a function of the solution of the free Schrödinger equation ρ_t^S (i.e., with $\lambda = \alpha = 0$), is

$$\begin{aligned}\langle q_1|\rho_t|q_2\rangle = {}& \frac{1}{2\pi\hbar}\int \mathrm{d}k \int \mathrm{d}y\, \mathrm{e}^{-(\mathrm{i}/\hbar)ky} F[k, q_1 - q_2, t] \\ & \times \left\langle y + \frac{q_1 + q_2}{2} + \frac{q_1 - q_2}{2}\,\mathrm{e}^{-2\lambda\alpha t} + \frac{kt}{2m}\left(1 - \frac{\Gamma_t}{2\lambda\alpha t}\right)\right| \\ & \times \rho_t^S \left| y + \frac{q_1 + q_2}{2} - \frac{q_1 - q_2}{2}\,\mathrm{e}^{-2\lambda\alpha t} - \frac{kt}{2m}\left(1 - \frac{\Gamma_t}{2\lambda\alpha t}\right)\right\rangle, \end{aligned} \quad (54)$$

with

$$F[k,x,t] = \exp\left\{-\frac{\lambda\alpha^2}{2\hbar^2}k^2 t + \frac{1}{8\alpha\Gamma_t^2}\left[\left(x\,\mathrm{e}^{-2\lambda\alpha t} - \frac{\Gamma_t}{2m\lambda\alpha}k\right)^2 K_1(t)\right.\right.$$
$$\left.\left. + 2x\left(x\,\mathrm{e}^{-2\lambda\alpha t} - \frac{\Gamma_t}{2m\lambda\alpha}k\right)K_2(t) + x^2 K_3(t)\right]\right\} \tag{55}$$

and

$$\begin{aligned} K_1(t) &= \Gamma_t^2 + 2\Gamma_t - 4\lambda\alpha t,\\ K_2(t) &= \mathrm{e}^{-4\lambda\alpha t} + 4\lambda\alpha t\,\mathrm{e}^{-2\lambda\alpha t} - 1,\\ K_3(t) &= -4\lambda\alpha t\,\mathrm{e}^{-4\lambda\alpha t} - \Gamma_t^2 + 2\Gamma_t\,\mathrm{e}^{-2\lambda\alpha t}, \end{aligned} \tag{56}$$

where we have defined $\Gamma_t = 1 - \mathrm{e}^{-2\lambda\alpha t}$. Taking the diagonal matrix elements, one has

$$p_t(x) = \frac{1}{2\pi\hbar}\int \mathrm{d}k \int \mathrm{d}y\,\mathrm{e}^{-(\mathrm{i}/\hbar)ky} F[k,0,t]$$
$$\times\left\langle y + x + \frac{kt}{2m}\left(1 - \frac{\Gamma_t}{2\lambda\alpha t}\right)\right|\rho_t^S\left|y + x - \frac{kt}{2m}\left(1 - \frac{\Gamma_t}{2\lambda\alpha t}\right)\right\rangle \tag{57}$$

and

$$F[k,0,t] = \exp\left\{-\frac{\lambda\alpha^2}{2\hbar^2}k^2 t + \frac{1}{32m^2\lambda^2\alpha^3}k^2 K_1(t)\right\}. \tag{58}$$

According to the values (7) and (8) for λ_0 and α_0, the above expressions can be expanded for small[13] $\lambda\alpha t$, leading to

$$p_t(x) \simeq \frac{1}{2\pi\hbar}\int \mathrm{d}k \int \mathrm{d}y\,\mathrm{e}^{-(\mathrm{i}/\hbar)ky}\exp\left\{-\left[\frac{\lambda k^2}{6m^2}t^3 + \frac{\lambda\alpha^2 k^2}{2\hbar^2}t\right]\right\}$$
$$\times\left\langle y + x + \frac{\lambda\alpha k t^2}{2m}\right|\rho_t^S\left|y + x - \frac{\lambda\alpha k t^2}{2m}\right\rangle. \tag{59}$$

We now focus on the case of a macroscopic object (let us say $m \geqslant 1$ g); one can further approximate the above expression by noting that for such values of m the exponential factor appearing in equation (59) damps all matrix elements such that the term $\lambda\alpha k t^2/2m$ is not vanishingly small; e.g. when $\lambda\alpha k t^2/2m \geqslant 10^{-15}$ m, then the second exponential in the above equation is much smaller than $\mathrm{e}^{-10^{15}(m/\mathrm{kg})}$. We can then neglect the two terms in the matrix elements and perform the integration over k, and we get

$$p_t(x) \simeq \sqrt{\frac{\beta_t}{\pi}}\int \mathrm{d}y\,\mathrm{e}^{-\beta_t y^2} p_t^S(x+y), \tag{60}$$

with

$$\beta_t = \frac{3}{2\hbar^2}\frac{m^2}{\lambda\left[1 + 3\left(\frac{m\alpha}{\hbar t}\right)^2\right]}\frac{1}{t^3}\begin{cases} \simeq 10^{43}\left(\dfrac{m}{\mathrm{kg}}\right)\left(\dfrac{\mathrm{s}}{t}\right)^3 & \text{for} \quad t \geqslant 10^{-11}\ \mathrm{s},\\ \geqslant 10^{65}\left(\dfrac{m}{\mathrm{kg}}\right)\left(\dfrac{\mathrm{s}}{t}\right) & \text{for} \quad t \leqslant 10^{-11}\ \mathrm{s}. \end{cases} \tag{61}$$

The exponent in (60) is extremely peaked with respect to the typical values the probability density $p_t^S(x)$ associated with the wavefunction of a macroscopic object takes, so that with very high accuracy we have

$$p_t(x) \simeq p_t^S(x). \tag{62}$$

[13] This means that we are considering only times $t \ll (\lambda\alpha)^{-1} \simeq 10^{20}$ s.

As discussed in [4], the probability measure $\mu_t(\Delta)$ which we have shown to be extremely close to the quantum probability obtained from the free Schrödinger equation can be interpreted as a probability measure for the collapse of the wavefunction of the macroscopic object within Δ, when Δ corresponds to an interval of the real axis of width greater or equal, e.g., to 10^{-5} cm [4].

To conclude, the previous analysis shows that under the above-listed conditions the probability for the wavefunction of a macro-object to be localized within an interval of the real axis is almost equal to the corresponding quantum probability as given by the Born rule.

7. Dynamics of microscopic and macroscopic systems

In this section, we discuss how our reduction model is related both to quantum and to classical mechanics. Our aim is to show that at the microscopic level the physical predictions of the model are almost identical to standard quantum predictions and that, at the same time, the model with high accuracy reproduces classical mechanics at the macro-level.

7.1. Micro-systems: comparison with standard quantum mechanics

Microscopic systems cannot be directly observed, and their properties can be analysed only by resorting to suitable measurement procedures. All physical predictions of our model, concerning the outcome of measurements, have the form $\mathbb{E}_{\mathbb{P}}[\langle\psi_t|S|\psi_t\rangle]$, where S is a suitable self-adjoint operator, typically a projection operator and it is easy to show that $\mathbb{E}_{\mathbb{P}}[\langle\psi_t|S|\psi_t\rangle] \equiv \mathrm{Tr}[S\rho_t]$, where the statistical operator $\rho_t \equiv \mathbb{E}_{\mathbb{P}}[|\psi_t\rangle\langle\psi_t|]$ satisfies equation (10). Accordingly, as already discussed in section 4, the testable effects of the stochastic process on the wavefunction are similar to the effect induced by a quantum environment on the particle, when both friction and diffusion are taken into account.

With our choice (7) and (8) for λ_0 and α_0, the testable effects of the stochastic process are of the same order of magnitude of those induced by the interaction of the system with particles and radiation of the *intergalactic space* [25]: such effects are very small and masked by most other sources of decoherence, so that they can be tested only by resorting to sophisticated experiments [26, 27]. This implies that the physical predictions of our model are very close to standard quantum-mechanical predictions.

7.2. Macro-objects: comparison with classical mechanics

A macroscopic object is made of elementary constituents strongly interacting among each other and, according to our model, its dynamics is governed by the following stochastic differential equation, which is the straightforward generalization of equation (1) to a system of N particles:

$$\mathrm{d}\psi_t(\{x\}) = \left[-\frac{\mathrm{i}}{\hbar}H_{\mathrm{TOT}}\,\mathrm{d}t + \sum_{n=1}^{N}\sqrt{\lambda_n}(A_n - r_{nt})\,\mathrm{d}W_t^n - \sum_{n=1}^{N}\frac{\lambda_n}{2}\left(A_n^\dagger A_n - 2r_{nt}A_n + r_{nt}^2\right)\mathrm{d}t\right]\psi_t(\{x\}); \tag{63}$$

the symbol $\{x\} \equiv x_1, x_2, \ldots, x_N$ represents the N spatial coordinates of the configuration space of the composite system, $W_t^1, W_t^2, \ldots, W_t^N$ are N independent Wiener processes, $r_{nt} = \langle\psi_t|\left[A_n + A_n^\dagger\right]|\psi_t\rangle/2$ and the localization operators A_n are given by expressions (4), with q

replaced by q_n, the position operator of the nth particle, and p replaced by p_n, the corresponding operator. Furthermore,

$$\lambda_n = \frac{m_n}{m_0}\lambda_0, \qquad \alpha_n = \frac{m_0}{m_n}\alpha_0 \tag{64}$$

and

$$H_{\text{TOT}} = H^0_{\text{TOT}} + \sum_{n=1}^{N} \frac{\lambda_n \alpha_n}{2}\{q_n, p_n\}, \tag{65}$$

H_0 being the standard quantum Hamiltonian for the composite system.

As custom, we separate the motion of the centre of mass from the internal motion. To this end, let

$$Q = \frac{1}{M}\sum_{n=1}^{N} m_n q_n \qquad \left(M = \sum_{n=1}^{N} m_n\right) \tag{66}$$

be the position operator associated with the centre-of-mass coordinate R and $\tilde{q}_n$ be the position operators associated with the internal coordinates $\tilde{x}_n = x_n - R$ $(n = 1, \ldots, N)$; let also P and $\tilde{p}_n$ be the corresponding momentum operators. Then, if H^0_{TOT} can be written as the sum of a term H^0_{CM} associated with the centre of mass and a term H^0_{REL} associated with the internal motion, it is easy to prove that the dynamics of the two types of degrees of freedom decouple; in particular, the equation for the centre of mass—the only one we consider here—becomes

$$\begin{aligned} \mathrm{d}\psi_t(R) = \Bigg[&-\frac{\mathrm{i}}{\hbar} H_{\text{CM}}\,\mathrm{d}t + \sqrt{\lambda_{\text{CM}}}(A_{\text{CM}} - r_{\text{CM},t})\,\mathrm{d}W_t \\ &- \frac{\lambda_{\text{CM}}}{2}\left(A^\dagger_{\text{CM}} A_{\text{CM}} - 2r_{\text{CM},t} A_{\text{CM}} + r^2_{\text{CM},t}\right)\mathrm{d}t\Bigg]\psi_t(R), \end{aligned} \tag{67}$$

with

$$H_{\text{CM}} = H^0_{\text{CM}} + \frac{\lambda_{\text{CM}}\alpha_{\text{CM}}}{2}\{Q, P\}, \tag{68}$$

$$r_{\text{CM},t} = \frac{1}{2}\langle\psi_t|\left[A^\dagger_{\text{CM}} + A_{\text{CM}}\right]|\psi_t\rangle, \tag{69}$$

$$A_{\text{CM}} = Q + \mathrm{i}\frac{\alpha_{\text{CM}}}{\hbar}P \tag{70}$$

and

$$W_t = \sum_{n=1}^{N}\sqrt{\frac{m_n}{M}}\,W^n_t, \tag{71}$$

which is easily proven to be a standard Wiener process. The two constants λ_{CM} and α_{CM} are defined by equations (6), with m equal to the total mass M of the composite system. Note that the separation of the centre-of-mass motion from the relative motion, for the non-Schrödinger terms of equation (63), is possible because of the specific dependence of the parameters λ_n and α_n on the masses m_n of the particles as given by equation (64).

According to equation (67), the centre of mass behaves like a particle whose dynamics, in the free case, has been discussed in detail in sections 5 and 6; we now show how the large numerical value for M, typical of macroscopic objects, affects the time evolution of the centre-of-mass wavefunction.

7.2.1. Collapse rate. According to equation (51), and assuming that the wavefunction is not already localized in space, i.e., that $\sigma_q(t) \gg \overline{\sigma}_q$, one has

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\sigma_O^2(t)]\right| \geqslant \frac{\lambda\hbar^2}{\overline{\sigma}_q^{\,4}}\sigma_q^4(t) = \left(\frac{2\lambda_0\omega^2\sin^2\theta}{m_0}\right)m^3\sigma_q^4(t), \tag{72}$$

with $2\lambda_0\omega^2\sin^2\theta/m_0 \simeq 10^{15}\ \mathrm{kg}^{-1}\ \mathrm{m}^{-2}\ \mathrm{s}^{-3}$. In the microscopic case, the rhs of equation (72) is in general small and negligible as we expect it to be, since the reduction mechanism must be ineffective on microscopic systems; in the macroscopic case instead it rapidly becomes big, due to the large value of the mass m, ensuring that *any* initial wavefunction rapidly converges towards a localized solution. We can then assume that, possibly after a very short transient period, the wavefunction describing the motion of the centre of mass of any macro-object is practically localized in space.

7.2.2. Behaviour of the stationary solution for macroscopic objects. We now discuss the time evolution of a typical localized wavefunction, i.e., a Gaussian stationary solution. Equations (36) and (37) imply that the two maxima in position and momentum of such wavefunctions fluctuate around their mean values; we now show that in the macroscopic regime these fluctuations are extremely small.

As a matter of fact, equations (42)–(44) imply for a stationary solution

$$C_{q^2}(t) = \lambda\ell^2 t - \frac{\hbar\ell}{2\lambda\alpha^2 m}\left(\frac{\cos\theta-\kappa}{\sin\theta}\right)(1-\mathrm{e}^{-2\lambda\alpha t}) + \frac{\hbar^2}{16\lambda^2\alpha^3 m^2}\left(\frac{\cos\theta-\kappa}{\sin\theta}\right)^2(1-\mathrm{e}^{-4\lambda\alpha t}) \tag{73}$$

$$C_{p^2}(t) = \frac{\hbar^2}{4\alpha}\left(\frac{\cos\theta-\kappa}{\sin\theta}\right)^2(1-\mathrm{e}^{-4\lambda\alpha t}), \tag{74}$$

where ℓ is defined as follows:

$$\ell = -\frac{\hbar}{2\lambda\alpha m}\left(\frac{\cos\theta-\kappa}{\sin\theta}\right) - \left(\frac{\sqrt{2}\hbar}{m\omega\sin\theta} - \alpha\right). \tag{75}$$

Since the exponential factors decay very slowly (we remember that $\lambda\alpha \simeq 10^{-20}\ \mathrm{s}^{-1}$), it is physically significant to consider only the linear term of their Taylor expansion; one then gets

$$C_{q^2}(t) \simeq \frac{4\hbar^2\lambda_0}{m_0\omega^2}\frac{t}{m} \simeq 10^{-33}\left(\frac{\mathrm{kg}}{m}\right)\left(\frac{t}{\mathrm{s}}\right)\mathrm{m}^2, \tag{76}$$

$$C_{p^2}(t) \simeq \frac{\hbar^2\lambda_0}{m_0}mt \simeq 10^{-43}\left(\frac{m}{\mathrm{kg}}\right)\left(\frac{t}{\mathrm{s}}\right)\frac{\mathrm{kg}^2\ \mathrm{m}^2}{\mathrm{s}^2}, \tag{77}$$

which are very small quantities, when m is the mass of a macro-object. Accordingly, in the macroscopic case, the actual values of the two peaks in position and momentum of a stationary Gaussian solution are very close to their average values which, as we have seen, evolve in time according to Newton's laws for a free particle moving in a (very weakly) dissipative medium. This proves that with high accuracy a stationary solution for the centre of mass of a macro-system practically behaves like a point moving in space according to the laws of classical mechanics.

Before concluding this section, we note that in equation (63), which describes the time evolution of many-particle systems, a different Wiener process is attached to each single particle; as such, the equation does not preserve the symmetry properties of the wavefunction for systems of identical particles. This limitation of the model can be overcome by reformulating it in the language of second quantization, in more or less the same way in which the GRW collapse model [1], which does not apply to identical particles, has been superseded by the CSL model [2]. This will be the subject of future research.

8. Time evolution of the mean energy

We now discuss one of the main purposes of our work, i.e., we show that within our collapse model the energy of an isolated system does not increase with a constant (even if negligible) rate, but reaches an asymptotic finite value. Indeed, this result is entailed by equation (10) for the statistical operator, which motivated our choice for the localization operator A.

As a matter of fact, the mean value $\mathbb{E}[\langle H_0\rangle_t]$ of the energy satisfies the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\langle H_0\rangle_t] = \frac{\lambda\hbar^2}{2m} - 4\lambda\alpha\mathbb{E}[\langle H_0\rangle_t], \tag{78}$$

whose solution is

$$\mathbb{E}[\langle H_0\rangle_t] = \left(E_0 - \frac{\hbar^2}{8m\alpha}\right)\mathrm{e}^{-4\lambda\alpha t} + \frac{\hbar^2}{8m\alpha}. \tag{79}$$

As we see, the mean energy of an isolated system changes in time and thus is not conserved; anyway it does not diverge for $t \to +\infty$, but reaches the asymptotic finite value

$$\lim_{t\to+\infty} \mathbb{E}[\langle H_0\rangle_t] = \frac{\hbar^2}{8m\alpha} = \frac{\hbar^2}{8m_0\alpha_0}, \tag{80}$$

corresponding to a temperature

$$T = \frac{\hbar^2}{4m_0\alpha_0 k_B} \simeq 10^{-1}\ \mathrm{K}, \tag{81}$$

which is independent of the mass of the particle. Note that the time evolution of $\mathbb{E}[\langle H_0\rangle_t]$ is very slow, the rate of change being equal to $4\lambda\alpha = 4\lambda_0\alpha_0 \simeq 10^{-20}\ \mathrm{s}^{-1}$ which means that, with very high accuracy, the mean energy remains constant for very long times.

The above equations imply that the stochastic process acts like a dissipative medium which, due to friction, slowly thermalizes all systems to the temperature T by absorbing or transferring energy to them depending on their initial states. Note that according to equation (81), a very 'cold' medium is enough to guarantee the localization in space of the wavefunctions of macroscopic objects. Note also that one recovers a GRW-type equation by setting $\alpha \to 0$, which corresponds to the high temperature limit $T \to +\infty$. This implies that the reason why in the original GRW reduction model [1] the energy increases and eventually diverges is that the noise acts like an infinite-temperature stochastic medium. Our model also shows that the increase of the mean energy of a quantum system subject to spontaneous localizations is not an intrinsic feature of space-collapse models (indeed, according to our model, the mean energy *decreases* for typical quantum systems) and it can be (partly) avoided with a suitable choice of the localizations operators.

The above discussion suggests that the model can be further developed by promoting W_t to a *real physical* medium with its own equations of motions, having a stochastic behaviour to which a temperature T can be associated and such that, with good accuracy, can be treated like a Wiener process. This would imply not only that the medium acts on the wavefunction by changing its state according to equation (1), but also that the wavefunction acts back on the medium according to equations which still have to be studied. The above suggestion opens the way to the possibility that by taking into account the energy of both the quantum system *and* the stochastic medium, one can restore perfect energy conservation not only on the average but also for single realizations of the stochastic process. A similar proposal has been considered by Pearle [28] and by Adler [9].

The price to pay in order to temper the energy non-conservation is that also the momentum is not conserved, not only for single realizations of the stochastic process (as it happens for all other collapse models) but also in the average, as equation (40) shows: the average momentum of any physical system slowly decays in time and asymptotically goes to zero. By the analogy with the quantum Brownian motion, the reason for this behaviour is quite simple: in order to thermalize a system to the temperature of the bath, momentum is exchanged between the system and the bath, which implies that the momentum of the quantum system alone is not conserved.

As suggested previously, one possibility to restore momentum conservation, as well as energy conservation, is to promote the stochastic field to a real physical field: by considering the energy and momentum of both systems, it could be possible that the two principles of energy and momentum conservation can be preserved.

These ideas will be subject of future research; we conclude by noting that, whatever its nature can be, the stochastic medium cannot be quantum in the usual sense since its coupling to the particle is not a standard coupling between two quantum systems: equation (1), in fact, is *not* the standard Schrödinger equation with a stochastic potential.

9. Conclusions

We have presented and analysed a collapse model which preserves the standard quantum-mechanical predictions and reproduces classical mechanics at the macroscopic level, at the same time avoiding the infinite growth of energy of the system, a criticized feature [15] of the space-collapse models that have appeared in the literature. This has been obtained by drawing on an analogy with quantum Brownian motion, where friction effects, directly related to diffusion for the preservation of the Heisenberg uncertainty relation, guarantee that the energy of the test particle reaches a finite value depending on the parameters of the model. The model is also characterized by the fact that the related master equation complies with the general structure of translation-covariant generators of quantum-dynamical semigroups obtained by Holevo [22], so that symmetry under translation is correctly taken into account, as compulsory for dynamical reduction models, which should not introduce any preferred space location.

Needless to say, the exploited analogy with the quantum Brownian motion master equation, typically used for the description of dissipation and decoherence, should in no way induce confusion on the different nature of the two issues of decoherence and of the measurement or macro-objectification problem in quantum mechanics; as has been stressed also in recent publications [25, 29], decoherence does not provide a solution to the measurement problem. This important conceptual difference notwithstanding, dynamical reduction models and models of environmental decoherence both impinge on the same mathematical inventory, typically used in the theory of open quantum systems [30], so that results obtained in the one field can often be fruitfully exploited in the other.

An extension of this approach might be pursued in order to cope with the infinite energy growth also in the original GRW model of dynamical reductions [1], building on the analogy with the quantum linear Boltzmann equation [21].

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Appendix A. The mean rate of change of the variance σ_O

In order to derive equation (51), we first compute the average value of the time derivative of the second moments of the operators q and p and of their symmetrized correlation. Using the evolution equation (1), we obtain through Itô calculus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\sigma_q^2(t)\big] = \frac{2\mathbb{E}\big[\sigma_{q,p}^2(t)\big]}{m} - 4\lambda\mathbb{E}\big[\sigma_q^4(t)\big] + 4\alpha\lambda\mathbb{E}\big[\sigma_q^2(t)\big], \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\sigma_p^2(t)\big] = -4\lambda\mathbb{E}\big[\sigma_{q,p}^4(t)\big] - 4\alpha\lambda\mathbb{E}\big[\sigma_p^2(t)\big] + \lambda\hbar^2, \tag{A.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\sigma_{q,p}^2(t)\big] = \frac{\mathbb{E}\big[\sigma_p^2(t)\big]}{m} - 4\lambda\mathbb{E}\big[\sigma_{q,p}^2(t)\sigma_q^2(t)\big]. \tag{A.3}$$

Since a_t is constant for a stationary solution, the stationary values of variances and correlation are such that the right-hand sides of (A.1)–(A.3) vanish:

$$\frac{\overline{\sigma}_{q,p}^2}{m} - 2\lambda\overline{\sigma}_q^4 + 2\alpha\lambda\overline{\sigma}_q^2 = 0, \tag{A.4}$$

$$\overline{\sigma}_{q,p}^4 + \alpha\overline{\sigma}_p^2 - \frac{\hbar^2}{4} = 0, \tag{A.5}$$

$$\frac{\overline{\sigma}_p^2}{m} - 4\lambda\overline{\sigma}_{q,p}^2\overline{\sigma}_q^2 = 0. \tag{A.6}$$

Moreover, equation (35) can be rewritten as

$$\overline{\sigma}_q^2\overline{\sigma}_p^2 - \overline{\sigma}_{q,p}^4 = \frac{\hbar^2}{4}. \tag{A.7}$$

For convenience, we define the quantity X, Y and Z by the equations

$$\sigma_q^2(t) = \overline{\sigma}_q^2(1+X), \qquad \sigma_p^2(t) = \overline{\sigma}_p^2(1+Y), \qquad \sigma_{q,p}^2(t) = \overline{\sigma}_{q,p}^2(1+Z),$$

so that equation (48) becomes

$$\sigma_O^2(t) = \overline{\sigma}_p^2(X+Y) - \frac{2\overline{\sigma}_{q,p}^4}{\overline{\sigma}_q^2}Z, \tag{A.8}$$

where we also made use of equation (A.7).

Using equations (A.1)–(A.3), we obtain for the average value of the time derivative of σ_O^2,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\sigma_O^2(t)\big] &= w_1\mathbb{E}[X] + w_2\mathbb{E}[Y] + w_3\mathbb{E}[Z]\\ &\quad - 4\lambda\big(\overline{\sigma}_q^2\overline{\sigma}_p^2\mathbb{E}[X^2] + \overline{\sigma}_{q,p}^4\mathbb{E}[Z^2] - 2\overline{\sigma}_{q,p}^4\mathbb{E}[XZ]\big),\end{aligned} \tag{A.9}$$

with

$$w_1 = -8\lambda\left(\overline{\sigma}_q^2\overline{\sigma}_p^2 - \frac{1}{2}\alpha\overline{\sigma}_p^2 - \overline{\sigma}_{q,p}^4\right), \tag{A.10}$$

$$w_2 = -2\left(2\alpha\lambda\overline{\sigma}_p^2 + \frac{\overline{\sigma}_{q,p}^2}{m}\frac{\overline{\sigma}_p^2}{\overline{\sigma}_q^2}\right), \tag{A.11}$$

$$w_3 = 2\frac{\overline{\sigma}_{q,p}^2}{m}\frac{\overline{\sigma}_p^2}{\overline{\sigma}_q^2}. \tag{A.12}$$

From equations (A.7) and (A.5), we get for w_1

$$w_1 = -4\lambda\overline{\sigma}_q^2\overline{\sigma}_p^2, \tag{A.13}$$

while using equation (A.4), one can prove that

$$w_2 = w_1. \tag{A.14}$$

Finally, from equation (A.6), we find that w_3 and w_1 are related as

$$w_3 = -\frac{2\overline{\sigma}_{q,p}^4}{\overline{\sigma}_q^2\overline{\sigma}_p^2}w_1, \tag{A.15}$$

so that we have

$$w_1X + w_2Y + w_3Z = w_1\left(X + Y - \frac{2\overline{\sigma}_{q,p}^4}{\overline{\sigma}_q^2\overline{\sigma}_p^2}Z\right) = -4\lambda\overline{\sigma}_q^2\sigma_O^2(t), \tag{A.16}$$

where the last equality is obtained through equations (A.8) and (A.13). Finally, the above result together with equation (A.7) allows us to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\sigma_O^2(t)\big] = -4\lambda\mathbb{E}\left[\overline{\sigma}_q^2\sigma_O^2(t) + \overline{\sigma}_{q,p}^4(X-Z)^2 + \frac{\hbar^2}{4}X^2\right], \tag{A.17}$$

which is just equation (51).

Appendix B. General solution of the master equation

We now show how to obtain the general solution of the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{\hbar}[H_0,\rho_t] - \frac{\lambda}{2}[q,[q,\rho_t]] - \frac{\lambda\alpha^2}{2\hbar^2}[p,[p,\rho_t]] - \mathrm{i}\frac{\lambda\alpha}{\hbar}[q,\{p,\rho_t\}], \tag{B.1}$$

given in (10), which has the same form of the quantum Brownian motion master equation, partially following the appendix of [25]. In particular, we want to express the general solution in the position representation $\langle q_1|\rho_t|q_2\rangle$ as a function of the solution of the pure Schrödinger equation $\langle q_1|\rho_t^S|q_2\rangle$, in order to point out the corrections to the position probability density due to the non-linear stochastic modification of the Schrödinger dynamics.

As a first step, we want to express the solution of (B.1) as a function of the generic initial condition ρ_0 according to

$$\langle q_1|\rho_t|q_2\rangle = \int \mathrm{d}q_{10}\,\mathrm{d}q_{20}G(q_1,q_2,t;q_{10},q_{20},0)\langle q_{10}|\rho_0|q_{20}\rangle, \tag{B.2}$$

where G is the Green function solution of the partial differential equation associated with (B.1) in the position representation satisfying the following initial condition:

$$G(q_1,q_2,t;q_{10},q_{20},0) \underset{t\to 0}{\longrightarrow} \delta(q_1-q_{10})\delta(q_2-q_{20}). \tag{B.3}$$

Once G is known, one may immediately express the general solution as a function of the solution of the Schrödinger equation by means of the free propagator G_0

$$\langle q_1|\rho_t|q_2\rangle = \int \mathrm{d}q_{10}\,\mathrm{d}q_{20}\int \mathrm{d}u\,\mathrm{d}v G(q_1,q_2,t;q_{10},q_{20},0)G_0(q_{10},q_{20},0;u,v,t)\langle u|\rho_t^S|v\rangle. \tag{B.4}$$

For calculational purposes, it is however convenient to consider the quantity

$$\tilde{\rho}_t(k,x) = \mathrm{Tr}\big(\rho_t\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(kq+xp)}\big), \tag{B.5}$$

corresponding to the characteristic function associated with the Wigner function. The master equation (B.1) thus becomes

$$\frac{\partial}{\partial t}\tilde{\rho}_t(k,x) = \frac{k}{m}\frac{\partial}{\partial x}\tilde{\rho}_t(k,x) - \frac{\lambda}{2}x^2\tilde{\rho}_t(k,x) - \frac{\lambda\alpha^2}{2\hbar^2}k^2\tilde{\rho}_t(k,x) - 2\lambda\alpha x\frac{\partial}{\partial x}\tilde{\rho}_t(k,x), \tag{B.6}$$

while (B.2) and (B.4) take the form

$$\tilde{\rho}_t(k,x) = \int \mathrm{d}k_0\,\mathrm{d}x_0 \tilde{G}(k,x,t;k_0,x_0,0)\tilde{\rho}_0(k_0,x_0) \tag{B.7}$$

and

$$\tilde{\rho}_t(k,x) = \int \mathrm{d}k_0\,\mathrm{d}x_0 \int \mathrm{d}r\,\mathrm{d}s \tilde{G}(k,x,t;k_0,x_0,0)\tilde{G}_0(k_0,x_0,0;r,s,t)\tilde{\rho}^S_t(r,s), \tag{B.8}$$

respectively, where $\tilde{\rho}^S_t(r,s)$ is again the solution of the free Schrödinger equation, $\tilde{G}_0$ is simply given by

$$\tilde{G}_0(k,x,t;k_0,x_0,0) = \delta(k-k_0)\delta\left(x - x_0 + \frac{k_0}{m}t\right) \tag{B.9}$$

and $\tilde{G}$ satisfies the following initial condition:

$$\tilde{G}(k,x,t;k_0,x_0,0) \underset{t\to 0}{\longrightarrow} \delta(k-k_0)\delta(x-x_0). \tag{B.10}$$

Equation (B.8) can be brought back to (B.4) by exploiting the inverse relation of equation (B.5), i.e.,

$$\langle q_1|\rho_t|q_2\rangle = \frac{1}{2\pi\hbar}\int \mathrm{d}k\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}k(\frac{q_1+q_2}{2})}\tilde{\rho}_t(k, q_1-q_2). \tag{B.11}$$

The key observation in order to determine $\tilde{G}$ is the fact that equation (B.1) preserves Gaussian states, so that given the Gaussian ansatz

$$\tilde{\rho}_t(k,x) = \exp\{-c_1k^2 - c_2kx - c_3x^2 - \mathrm{i}c_4k - \mathrm{i}c_5x - c_6\}, \tag{B.12}$$

one easily obtains the evolved state by expressing the coefficients at time t, $c_i(t)$, as a function of the initial state characterized by the values of the coefficient at time zero. Considering an initial Gaussian state $\tilde{\rho}_0^{k_0x_0,\epsilon\eta}(k,x)$ which in the limit of small ϵ and η approximates the Dirac delta, according to

$$\tilde{\rho}_0^{k_0x_0,\epsilon\eta}(k,x) \underset{\epsilon,\eta\to 0}{\longrightarrow} \delta(k-k_0)\delta(x-x_0), \tag{B.13}$$

one has

$$\tilde{G}(k,x,t;k_0,x_0,0) = \lim_{\epsilon,\eta\to 0}\tilde{\rho}_t^{k_0x_0,\epsilon\eta}(k,x). \tag{B.14}$$

Coming back to the ansatz (B.12), the coefficients satisfy the equations

$$\begin{aligned}
&\dot{c}_1(t) = \frac{c_2(t)}{m} + \frac{\lambda\alpha^2}{2\hbar^2}, \qquad \dot{c}_2(t) = \frac{2c_3(t)}{m} - 2\lambda\alpha c_2(t),\\
&\dot{c}_3(t) = \frac{\lambda}{2} - 4\lambda\alpha c_3(t), \qquad \dot{c}_4(t) = \frac{c_5(t)}{m}, \qquad \dot{c}_5(t) = -2\lambda\alpha c_5(t),
\end{aligned} \tag{B.15}$$

with solutions

$$
\begin{aligned}
c_1(t) &= c_1(0) + c_2(0)\frac{\Gamma_t}{2m\lambda\alpha} + c_3(0)\frac{\Gamma_t^2}{4m^2\lambda^2\alpha^2} - \frac{1}{32m^2\lambda^2\alpha^3}\left(\Gamma_t^2 + 2\Gamma_t - 4\lambda\alpha t\right) + \frac{\lambda\alpha^2}{2\hbar^2}t,\\
c_2(t) &= c_2(0)\,\mathrm{e}^{-2\lambda\alpha t} + c_3(0)\frac{\Gamma_t\,\mathrm{e}^{-2\lambda\alpha t}}{m\lambda\alpha} + \frac{\Gamma_t^2}{8m\lambda\alpha^2},\\
c_3(t) &= \frac{1}{8\alpha} + \left(c_3(0) - \frac{1}{8\alpha}\right)\mathrm{e}^{-4\lambda\alpha t},\\
c_4(t) &= c_4(0) + c_5(0)\frac{\Gamma_t}{2m\lambda\alpha},\\
c_5(t) &= c_5(0)\,\mathrm{e}^{-2\lambda\alpha t},
\end{aligned}
\tag{B.16}
$$

with $\Gamma_t = 1 - \mathrm{e}^{-2\lambda\alpha t}$ and c_6 simply a constant. For the choice

$$
\tilde{\rho}_0^{k_0 x_0,\epsilon\eta}(k,x) = \frac{1}{\pi\sqrt{\epsilon\eta}}\exp\left\{-\frac{1}{\epsilon}(k-k_0)^2\right\}\exp\left\{-\frac{1}{\eta}(x-x_0)^2\right\},
\tag{B.17}
$$

one has, exploiting (B.16),

$$
\begin{aligned}
\tilde{\rho}_t^{k_0 x_0,\epsilon\eta}(k,x) &= \frac{1}{\pi\sqrt{\epsilon\eta}}\exp\left\{-\frac{1}{\epsilon}(k-k_0)^2\right\}\exp\left\{-\frac{1}{\eta}\left[\frac{\Gamma_t k}{2m\lambda\alpha} - (x_0 - x\,\mathrm{e}^{-2\lambda\alpha t})\right]^2\right\}\\
&\quad\times\exp\left\{-\frac{\lambda\alpha^2}{2\hbar^2}k^2 t\right\}\exp\left\{\frac{1}{8\alpha}\left[k^2\frac{K_1(t)}{4m^2\lambda^2\alpha^2} - kx\frac{\Gamma_t^2}{m\lambda\alpha} - x^2(1-\mathrm{e}^{-4\lambda\alpha t})\right]\right\},
\end{aligned}
\tag{B.18}
$$

so that taking the limit one has

$$
\begin{aligned}
\tilde{G}(k,x,t;k_0,x_0,0) &= \delta(k-k_0)\,\delta\left(\frac{\Gamma_t k}{2m\lambda\alpha} - (x_0 - x\,\mathrm{e}^{-2\lambda\alpha t})\right)\\
&\quad\times\exp\left\{-\frac{\lambda\alpha^2}{2\hbar^2}k^2 t\right\}\exp\left\{\frac{1}{8\alpha\Gamma_t^2}\left[x_0^2 K_1(t) + 2x x_0 K_2(t) + x^2 K_3(t)\right]\right\},
\end{aligned}
\tag{B.19}
$$

with

$$
\begin{aligned}
K_1(t) &= \Gamma_t^2 + 2\Gamma_t - 4\lambda\alpha t,\\
K_2(t) &= \mathrm{e}^{-4\lambda\alpha t} + 4\lambda\alpha t\,\mathrm{e}^{-2\lambda\alpha t} - 1,\\
K_3(t) &= -4\lambda\alpha t\,\mathrm{e}^{-4\lambda\alpha t} - \Gamma_t^2 + 2\Gamma_t\,\mathrm{e}^{-2\lambda\alpha t}
\end{aligned}
\tag{B.20}
$$

and therefore (B.8) now explicitly becomes

$$
\begin{aligned}
\tilde{\rho}_t(k,x) &= \exp\left\{-\frac{\lambda\alpha^2}{2\hbar^2}k^2 t\right\}\exp\left\{\frac{1}{8\alpha\Gamma_t^2}\left[\left(x\,\mathrm{e}^{-2\lambda\alpha t} + \frac{\Gamma_t k}{2m\lambda\alpha}\right)^2 K_1(t)\right.\right.\\
&\quad\left.\left. + 2x\left(x\,\mathrm{e}^{-2\lambda\alpha t} + \frac{\Gamma_t k}{2m\lambda\alpha}\right)K_2(t) + x^2 K_3(t)\right]\right\}\\
&\quad\times\tilde{\rho}_t^S\left(k, x\,\mathrm{e}^{-2\lambda\alpha t} + \frac{\Gamma_t k}{2m\lambda\alpha}\left(1 - \frac{2\lambda\alpha t}{\Gamma_t}\right)\right).
\end{aligned}
\tag{B.21}
$$

Exploiting the inversion formula (B.11) together with the expression

$$
\tilde{\rho}_t(k,x) = \int \mathrm{d}y\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}ky}\left\langle y + \frac{x}{2}\right|\rho_t\left|y - \frac{x}{2}\right\rangle
\tag{B.22}
$$

equivalent to (B.5), one finally obtains the desired explicit expression for (B.4):

$$
\begin{aligned}
\langle q_1|\rho_t|q_2\rangle = \frac{1}{2\pi\hbar}\int \mathrm{d}k \int \mathrm{d}y\, \mathrm{e}^{-(\mathrm{i}/\hbar)ky} \exp\left\{-\frac{\lambda\alpha^2}{2\hbar^2}k^2 t\right\} \\
\times \exp\left\{\frac{1}{8\alpha\Gamma_t^2}\left[\left((q_1-q_2)\,\mathrm{e}^{-2\lambda\alpha t} - \frac{\Gamma_t}{2m\lambda\alpha}k\right)^2 K_1(t)\right.\right. \\
\left.\left. + 2(q_1-q_2)\left((q_1-q_2)\,\mathrm{e}^{-2\lambda\alpha t} - \frac{\Gamma_t}{2m\lambda\alpha}k\right)K_2(t) + (q_1-q_2)^2 K_3(t)\right]\right\} \\
\times \left\langle y + \frac{q_1+q_2}{2} + \frac{q_1-q_2}{2}\,\mathrm{e}^{-2\lambda\alpha t} + \frac{kt}{2m}\left(1-\frac{\Gamma_t}{2\lambda\alpha t}\right)\right| \\
\times \rho_t^S \left| y + \frac{q_1+q_2}{2} - \frac{q_1-q_2}{2}\,\mathrm{e}^{-2\lambda\alpha t} - \frac{kt}{2m}\left(1-\frac{\Gamma_t}{2\lambda\alpha t}\right)\right\rangle. \qquad \text{(B.23)}
\end{aligned}
$$

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